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Nonlinear vibration of buckled beams: some exact solutions

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Abstract

In this paper, exact solutions are obtained for the dynamics of buckled beams with different types of end conditions. The differential equation for the vibrations of buckled beams, with restrictions on axial stretch, is nonlinear. Using the modes of the corresponding linear problem, which readily satisfy the boundary conditions, the natural frequencies are obtained for the nonlinear problem by using Jacobi elliptic functions. Analytical solutions for the dynamics of buckled beams, with various rotational restraints, are presented. © 2001 Published by Elsevier Science Ltd.

Keywords: Buckled beams; Vibrations; Exact solutions; Fourier series; Jacobi elliptic functions

1. Introduction

The subject of the vibration of buckled beams has been studied for many years. Burgreen (1951) obtained the amplitude dependent first natural frequency for a post-buckled simply supported column. Woinowsky-Krieger (1950) obtained the expression for amplitude dependent natural frequencies for simply supported beam, with an axial load that is below the value of the buckling load using elliptical functions. Bouwstra and Geijselaers (1991) presented an approximation for natural frequencies of fixed beam, subjected to axial loads, for buckled or initially deflected beams. In their work, the amplitudes of vibrations were assumed to be small; the nonlinearity was linearized by taking only the linear part of the deflections. Other approaches include the use of approximate methods like the Galerkin's method and perturbation methods. Min and Eisley (1980) considered a simply supported beam and assumed three mode shapes from the corresponding linear problem for nonlinear analysis using Galerkin's method. Tseng and Dugunji (1971) used the first two linear buckling modes to investigate the dynamic behavior of a buckled beam with fixed ends using Galerkin's method.

Nayfeh et al. (1995) and Chen (1994) perturbed static buckled shapes to obtain the modes and associated natural frequencies of buckled beams with fixed and simply supported boundary conditions. Their approach was to consider a beam that is statically loaded to the post-buckled load range, and disturb the

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beam by small deflections to obtain dynamic solutions. In contrast, one can consider a beam that is vibrating due to lateral loads or corresponding initial conditions. Then, an axial load can be applied and increased to the post-buckled load range. Thus, in this case, the beam is vibrating simultaneously under the combined action of lateral and axial loads.

In this paper, we consider such a beam that is subjected simultaneously to axial and lateral loads (or initial conditions) without first statically buckling the beam. If the neutral axis of the beam were allowed to stretch axially without any restraint, the axial stretching term or the nonlinear term is not present in the governing equation. In this case, the ends of the beam are not constrained. They can move closer to each other as the loading increases or the beam vibrates. Then, the vibration problem is a linear vibration problem. Eliminating the end constraints will reduce the stiffness of the beam. As a result, the natural frequencies will be lower. Instead of completely restraining the neutral axis of the beam, axially, we can consider axial springs. In this case, the vibration problem is a nonlinear vibration problem. In this paper, we also seek closed form exact solutions, instead of approximate solutions, to the problem of nonlinear vibrations of buckled beams, with nonlinearity due to axial stretch effects and with different boundary conditions. Here, the term exact solution is used to denote closed form solutions of the governing differential equation. The solution satisfies the nonlinear governing equation and the boundary conditions exactly.

In summary, we present exact solutions to the nonlinear problem of the vibration of buckled beam with elastic end restraints and axial stretch due to immovable ends. Vibrations of beams with general elastic end restraints have been studied by many researchers. This includes, works done by Rao and Raju (1978) for large amplitude vibrations, Laura and Gutirrez (1986) for a cantilever beam, Lee and Ke (1990) for nonuniform beams, and Maurizi et al. (1990) for Timoshenko beams. Rao and Naidu (1994) have also discussed the nonlinear behavior of the end supports. These works do not contain exact solutions presented in this paper. The deflection function for the nonlinear problem has been derived by using the corresponding linear mode shapes and Jacobi elliptic functions.

2. Analysis

2.1. Equations of motion

The differential equation of motion for a beam with an axial load P is

$$\mathrm{EI}\frac{\partial^4 w}{\partial x^4}+\mu\frac{\partial^2 w}{\partial^2 t}+P\frac{\partial^2 w}{\partial x^2}-\frac{\mathrm{EA}}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\,\frac{\partial^2 w}{\partial x^2}=F(x,t), \tag{1}$$

where EI is the flexural rigidity, EA, the axial rigidity, L, the length of the beam, μ, the mass per unit length, P, the axial load, and $F(x,t)$, the lateral loading function. The nonlinear term that appears in Eq. (1) is due to the interaction of the transverse deflection and the axial force caused by immovable supports. The beam may vibrate with large amplitudes but the curvatures are assumed to be small and Hooke's law is assumed to be valid. In general, the linear solution to the dynamic response of the beam is written as follows:

$$w=W+W_{\mathrm{p}}, \tag{2}$$

where W is the homogeneous solution corresponding to the free vibration of the beam, and W_{p} is the particular solution. For the problem with nonlinear effects, the solution cannot be calculated by superposition.

The boundary conditions at both ends can be written as

$$\text{at } x=0,\ \alpha_0 w_0=-\mathrm{EI}\partial^3 w/\partial x^3, \tag{3}$$

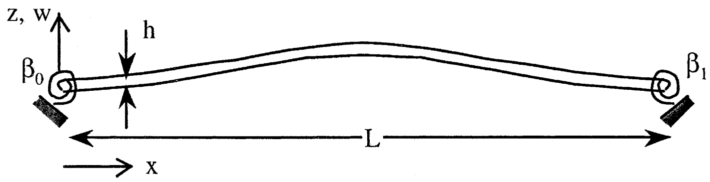

Fig. 1. Diagram of a typical buckled beam.

$$\beta_0 \partial w/\partial x = \mathrm{EI}\partial^2 w/\partial x^2 (x = 0), \tag{4}$$

$$\text{at } x = L, \ \ \alpha_1 w_1 = \mathrm{EI}\partial^3 w/\partial x^3, \tag{5}$$

$$\beta_1 \partial w/\partial x = -\mathrm{EI}\partial^2 w/\partial x^2 (x = L), \tag{6}$$

where α_0 and α_1 are the linear spring constants restraining the lateral deflections. Similarly, β_0 and β_1 are the rotational spring constants at $x = 0$ and L, respectively. In this study, the end supports are immovable, axially and laterally, therefore the linear springs are omitted. First, we consider the case of a fixed beam. Then, we consider general elastic end restraints. The diagram of a typical buckled beam with rotational end restraints is shown in Fig. 1.

2.2. Nonlinear vibration analysis for fixed beams by using conventional linear modes

In the published literature, no exact solutions are reported for the case of a fixed beam. Only solutions with perturbation to the static buckled shape are available. Therefore, as the first step, the exact solution for the nonlinear problem of vibrations of buckled beams with fixed ends is obtained using the conventional linear mode shapes and Jacobi elliptic functions. The linear modes shape for a fixed–fixed beam is (Harris and Crede, 1976)

$$\phi(x) = \cos\left(\frac{\gamma x}{L}\right) - \cosh\left(\frac{\gamma x}{L}\right) + R\left(\sin\left(\frac{\gamma x}{L}\right) - \sinh\left(\frac{\gamma x}{L}\right)\right), \tag{7}$$

where $R = (\sin(\gamma) + \sinh(\gamma))/(\cos(\gamma) - \cosh(\gamma))$ and γ is a constant that depends on the boundary conditions of the beam. For a beam with fixed–fixed ends, the first mode has a value of $\gamma_1 = 4.730$.

For the nonlinear case, the displacement function is assumed to be in the following form:

$$w(x,t) = \hat{a}\phi(x)\psi(t). \tag{8}$$

In this equation, $\psi(t)$ is an arbitrary function of time, $\phi(x)$ is the mode shape with $\gamma = \gamma_i$ defined by Eq. (7) and $\hat{a}$ is an arbitrary constant that represents the amplitude of the deflection. Substituting Eq. (8) and its derivatives into the governing equation (1) for the nonlinear problem, then multiplying by the mode shape and integrating in its domain we obtain

$$\ddot{\psi} + \left(\frac{\mathrm{EI}}{\mu}\Phi_4 + \frac{P}{\mu}\Phi_2\right)\psi - \frac{\mathrm{EI}}{2\mu L}\left(\frac{\hat{a}}{r}\right)^2 \Phi_1 \Phi_2 \psi^3 = 0, \tag{9}$$

where

$$\Phi_4 = \frac{\int_0^L \phi^{(\mathrm{iv})}\phi \,\mathrm{d}x}{\int_0^L \phi^2 \,\mathrm{d}x}, \qquad \Phi_2 = \frac{\int_0^L \phi''\phi \,\mathrm{d}x}{\int_0^L \phi^2 \,\mathrm{d}x}, \qquad \Phi_1 = \int_0^L (\phi')^2 \,\mathrm{d}x,$$

and r is the radius of gyration, defined as $r = \sqrt{I/A}$.

Integrating Eq. (9) once with respect to time, with the initial conditions at $t = 0$, $\Psi = 1$ and $\mathrm{d}\Psi/\mathrm{d}t = 0$, gives

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 = \chi_1(1-\psi^2) + \frac{\chi_2}{2}(1-\psi^4), \tag{10}$$

where

$$\chi_1 = \left(\frac{\mathrm{EI}}{\mu}\Phi_4 + \frac{P}{\mu}\Phi_2\right) \quad \text{and} \quad \chi_2 = -\frac{\mathrm{EI}}{2\mu L}\left(\frac{\hat{a}}{r}\right)^2 \Phi_1\Phi_2.$$

Since Φ_2 has a negative value, χ_2 is always positive. χ_1 is positive in the pre-buckled stage and becomes negative when the axial load exceeds the buckling load. In the post buckled range, solutions are possible only if $\chi_2 > \chi_1$. This means that the deflection amplitudes must be sufficiently large to make $\chi_1 + \chi_2$ positive. In other words, the nonlinear term should be large enough to provide a positive stiffness to have a solution in the post-buckled regime.

The parameter χ_1 and χ_2 are regrouped in the following way:

$$p^2 = \chi_1 + \chi_2, \qquad k^2 = \frac{\chi_2}{2p^2} \tag{11a, b}$$

such that the differential equation has solutions in terms of Jacobi elliptic function. Hence, Eq. (10) can be rewritten as follows:

$$\begin{aligned}\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 &= \chi_1 + \frac{\chi_2}{2} - \chi_1\psi^2 - \frac{\chi_2}{2}\psi^4 = p^2 - k^2p^2 - (p^2 - 2k^2p^2)\psi^2 - k^2p^2\psi^4 \\ &= p^2(1 - k^2 - (1-2k^2)\psi^2 - k^2\psi^4)\end{aligned}$$

and it reduces to

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}(pt)}\right)^2 = (1-\psi^2)(k^2\psi^2 - k^2 + 1). \tag{12}$$

Then, assuming $\psi = \cos\varphi$ we can obtain Jacobi elliptic function (Byrd and Friedman, 1971) with the modulus k, defined by Eq. (11b),

$$pt = \int_0^{\varphi} \frac{\mathrm{d}\varphi}{\sqrt{(1 - k^2\sin^2\varphi)}}. \tag{13}$$

From the inversion of Eq. (13), the solution for ψ can be obtain as follows:

$$\psi = \mathrm{cn}[pt, k]. \tag{14}$$

The Jacobi elliptic functions are real for real p and real k^2 between 0 and 1. For $k^2 = 0$ the function reduces to elementary trigonometric function $\cos pt$ with the period of 2π. In this case, the motion of the beam is harmonic. In other words, the nonlinear effects are not included, $\chi_2 = 0$. For $k^2 = 1$ the function reduces to elementary functions sech pt and the motion of the beam is aperiodic, approaching the straight position asymptotically with time. In this case, the external axial load, P, exceeds the buckling load of the beam and the axial load due to axial stretching cannot compensate this load. Therefore, when the axial load greater than the buckling load, large amplitude deflections are required in order to cause the beam to vibrate, such that $\chi_2 > \chi_1$.

The period of the function $\mathrm{cn}[pt, k]$ is $4K$ and is defined by using the complete elliptic integral,

$$4K = 4\int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}}. \tag{15}$$

Then, the corresponding frequency for this nonlinear problem for each mode is defined by using the following equation:

$$\tilde{\omega}_i = \frac{\pi\sqrt{\chi_1+\chi_2}}{2K}. \tag{16}$$

Using Eq. (8) we can write the solution for the free vibration of a buckled beam with fixed end restraints in the following form:

$$w(x,t) = \hat{a}\left\{\cos\left(\frac{\gamma x}{L}\right) - \cosh\left(\frac{\gamma x}{L}\right) + R\left(\sin\left(\frac{\gamma x}{L}\right) - \sinh\left(\frac{\gamma x}{L}\right)\right)\right\}\mathrm{cn}[pt, k] \tag{17}$$

with $\gamma = \gamma_i$ and $p = p_i$ for different modes.

2.3. Nonlinear vibration analysis for buckled beam with elastic restraints

For a beam with arbitrary elastic restraints, solutions are obtained by using finite Fourier transforms for the linear solutions. Jacobi elliptic functions are used for the nonlinear problem. The theory for linear free vibration, by using finite Fourier transforms, has been discussed by Wang and Lin (1997). Based on these linear mode shapes and Jacobi elliptic functions, solutions for the nonlinear dynamics of buckled beams are now developed.

2.3.1. Linear vibration analysis

The essential features of the theory for the linear free vibration are briefly summarized here for beams with rotational restraints and zero ends displacements ($w_0 = w_1 = 0$). The linear governing equation for a beam in free vibration is

$$\mathrm{EI}\frac{\partial^4 w}{\partial x^4} + \mu\frac{\partial^2 w}{\partial^2 t} = 0. \tag{18}$$

In this study, the deflection $w(x)$ and its derivatives are expressed in terms of Fourier sine series.

$$w(x,t) = \sum_{m=1}^{\infty} q_m(t)\sin\alpha_m x, \quad 0 < x < L, \tag{19}$$

where $\alpha_m = m\pi/L$ and $w = w_0$ and w_1 at $x = 0$ and L, respectively. The representation of the derivatives of w, based on the usual definitions of finite Fourier transform, becomes

$$\frac{\partial w(x,t)}{\partial x} = \frac{w_1 - w_0}{L} + \sum_{m=1}^{\infty}(a_m + \alpha_m q_m(t))\cos\alpha_m x, \quad 0 \leqslant x \leqslant L, \tag{20}$$

where $a_m = 2[(-1)^m w_1 - w_0]/L$,

$$\frac{\partial^2 w(x,t)}{\partial x^2} = -\sum_{m=1}^{\infty}\alpha_m(a_m + \alpha_m q_m(t))\sin\alpha_m x, \quad 0 < x < L, \tag{21}$$

with $\partial^2 w/\partial x^2 = B_0$ and B_1 at $x = 0$ and L, respectively,

$$\frac{\partial^3 w(x,t)}{\partial x^3} = \frac{B_1 - B_0}{L} + \sum_{m=1}^{\infty} \left\{ \frac{2}{L}[(-1)^m B_1 - B_0] - \alpha_m^2 (a_m + \alpha_m q_m(t)) \right\} \cos \alpha_m x, \quad 0 \leqslant x \leqslant L, \tag{22}$$

$$\frac{\partial^4 w(x,t)}{\partial x^4} = -\sum_{m=1}^{\infty} \alpha_m \left\{ \frac{2}{L}[(-1)^m B_1 - B_0] - \alpha_m^2 (a_m + \alpha_m q_m(t)) \right\} \sin \alpha_m x, \quad 0 < x < L. \tag{23}$$

Mathematical discussions and proofs on the procedure regarding differentiation of Fourier series and finite Fourier transform for a function defined on the interval $[0, L]$ can be found in Tolstov (1962).

These Eqs. (19)–(23) are substituted into the governing equation (18) by assuming a harmonic motion for the time variation,

$$q_m(t) = \bar{q}_m e^{j\omega t}, \qquad B_0 = b_0 e^{j\omega t}, \qquad B_1 = b_1 e^{j\omega t}, \tag{24a–c}$$

where ω is the natural frequency of the system. Then, the coefficient $\bar{q}_m$ can be expressed in terms of b_0 and b_1 as follows:

$$\bar{q}_m = \frac{2\omega_m^2 [b_0 - (-1)^m b_1]}{L\alpha_m^3 (\omega^2 - \omega_m^2)}, \tag{25}$$

where $\omega_m = \sqrt{EI/\mu}(m\pi/L)^2$ are the natural frequencies of the beam with simply supported ends. Hence, the displacement function for the free vibration of the beam having rotational restraint at both ends can be expressed as

$$w(x,t) = \sum_{m=1}^{\infty} \frac{2\omega_m^2 [b_0 - (-1)^m b_1]}{L\alpha_m^3 (\omega^2 - \omega_m^2)} \sin \alpha_m x e^{j\omega t}. \tag{26}$$

Applying the rotationally restrained boundary conditions (4) and (6) at $x = 0$ and L, one will have two simultaneous homogeneous equations:

$$a_{11} b_0 + a_{12} b_1 = 0, \tag{27}$$

$$a_{21} b_0 + a_{22} b_1 = 0, \tag{28}$$

where

$$a_{11} = 1/2 - \bar{\beta}_0 \bar{a}_{11}, \quad a_{22} = 1/2 - \bar{\beta}_1 \bar{a}_{22}, \quad a_{12} = \bar{\beta}_0 \bar{a}_{12}, \quad a_{21} = \bar{\beta}_0 \bar{a}_{21},$$

$$\bar{a}_{11} = \bar{a}_{22} = \sum_{m=1}^{\infty} \frac{m^2}{\lambda_i^4 - m^4}, \qquad \bar{a}_{12} = \bar{a}_{21} = \sum_{m=1}^{\infty} (-1)^m \frac{m^2}{\lambda_i^4 - m^4},$$

$$\lambda_i^4 = \frac{\mu}{EI} \left(\frac{L}{\pi} \right)^4 \omega^2, \quad \beta_0 = \frac{\bar{\beta}_0 L}{EI\pi^2}, \quad \beta_1 = \frac{\bar{\beta}_1 L}{EI\pi^2},$$

in which the a_{ij} contain ω as a parameter which can be obtained by requiring the determinant of the coefficient matrix of Eqs. (27) and (28) to vanish, i.e.,

$$a_{11} a_{22} - a_{12} a_{21} = 0. \tag{29}$$

After obtaining the natural frequency ω, the associated mode shape can be determined in terms of Fourier series using Eq. (26) and the relationship in Eq. (27) or Eq. (28). Hence, the solution for a beam with rotationally restrained supports is

$$w_i = A_i \sum_{m=1}^{\infty} \sigma_m \sin \alpha_m x \cos \omega_i t, \tag{30}$$

where

$$\sigma_m = \frac{m^2}{\lambda_i^4 - m^4}\left(1 + (-1)^m \frac{a_{11}}{a_{12}}\right) \quad \text{and} \quad A_i = \frac{2L^2}{\pi^3} b_0.$$

2.3.2. Nonlinear vibration analysis for elastically restrained boundary conditions

To solve the corresponding nonlinear vibration problem of buckled beams, we start with the modes of the linear problem in Eq. (30), which already satisfy all the boundary conditions. The solution to the nonlinear problem is approached as follows. Linear mode shapes determine the spatial variation of the displacement function and satisfy the boundary conditions:

$$w = \hat{a} \sum_{m=1}^{\infty} \sigma_m \sin \alpha_m x \psi(t), \tag{31}$$

where $\psi(t)$ is an arbitrary unknown function of time, σ_m, $\hat{a}$ is an arbitrary constant that represents the modal amplitude and σ_m defines the mode shape of the linear problem as described in Eq. (30). Substituting Eq. (31) and its derivatives into the governing equation (1) for the nonlinear problem, we obtain

$$\ddot{\psi} + \left(\frac{\sum_{m=1}^{\infty} \omega_m^2 \sigma_m^2}{\sum_{m=1}^{\infty} \sigma_m^2} - \frac{P}{P_{\rm cr}} \frac{\sum_{m=1}^{\infty} \frac{\omega_m^2 \sigma_m^2}{m}}{\sum_{m=1}^{\infty} \sigma_m^2} + \frac{2L^2}{\hat{a}\pi^3} \frac{\sum_{m=1}^{\infty} \frac{\omega_m^2 \sigma_m^2}{m^3}}{\sum_{m=1}^{\infty} \sigma_m^2} [b_0 - (-1)^m b_1] \right) \psi + \frac{\rm EI}{4\mu} \left(\frac{\hat{a}}{r} \right)^2 \frac{\left(\sum_{m=1}^{\infty} \omega_m^2 \sigma_m^2 \right)^2}{\sum_{m=1}^{\infty} \sigma_m^2} \psi^3 = 0. \tag{32}$$

By using the known relationship between b_0 and b_1 (Eq. (27)), Eq. (32) can be rewritten as

$$\ddot{\psi} + \left(\frac{\sum_{m=1}^{\infty} \omega_i^2 \sigma_m^2}{\sum_{m=1}^{\infty} \sigma_m^2} - \frac{P}{P_{\rm cr}} \frac{\sum_{m=1}^{\infty} \frac{\omega_m^2 \sigma_m^2}{m}}{\sum_{m=1}^{\infty} \sigma_m^2} \right) \psi + \frac{\omega_1^2}{4} \left(\frac{\hat{a}}{r} \right)^2 \frac{\left(\sum_{m=1}^{\infty} m^2 \sigma_m^2 \right)^2}{\sum_{m=1}^{\infty} \sigma_m^2} \psi^3 = 0. \tag{33}$$

As discussed in the Section 2.2, we can solve Eq. (33) using Jacobi elliptic functions with the notations p and k as follows:

$$p^2 = \chi_1 + \chi_2, \qquad k^2 = \frac{\chi_2}{2p^2}, \tag{34a, b}$$

$$\chi_1 = \left(\frac{\sum_{m=1}^{\infty} \omega_i^2 \sigma_m^2}{\sum_{m=1}^{\infty} \sigma_m^2} - \frac{P}{P_{\rm cr}} \frac{\sum_{m=1}^{\infty} \frac{\omega_m^2 \sigma_m^2}{m}}{\sum_{m=1}^{\infty} \sigma_m^2} \right), \qquad \chi_2 = \frac{\omega_1^2}{4} \left(\frac{\hat{a}}{r} \right)^2 \frac{\left(\sum_{m=1}^{\infty} m^2 \sigma_m^2 \right)^2}{\sum_{m=1}^{\infty} \sigma_m^2}. \tag{34c, d}$$

Hence, using Eq. (31) we obtain the solution for the nonlinear vibration of a buckled beam with arbitrary rotational end restraints in the form of

$$w = \hat{a} \sum_{m=1}^{\infty} \sigma_m {\rm cn}[pt, k] \sin \alpha_m x. \tag{35}$$

The corresponding frequency for this nonlinear problem for each mode i is defined by the following equation:

$$\tilde{\omega}_i = \frac{\pi \sqrt{\chi_1 + \chi_2}}{2K}. \tag{36}$$

3. Numerical results

The first three natural frequencies for the beam having different end restraints are discussed in this section. In particular, we consider a glass-epoxy composite beam with the following dimensions: 25 mm in width, 3.2 mm in thickness and 275 mm in length. The composite was made from woven glass fiber with epoxy with the elastic modulus of $20\,947 \times 10^6$ N/m^2 and mass of 45.7 g. Variations of frequency with axial load and deflection amplitude for various end restraints are presented. The parameters are nondimensionalized as follows, the nonlinear frequency is nondimensionalized with respect to its corresponding linear frequency, $\tilde{\omega}/\omega_0(f/f_0)$, the axial load with respect to its corresponding critical buckling load, P/P_{cr}, and the amplitude with respect to the radius of gyration, a/r.

3.1. Comparison with existing results

For a simply supported beam, some results are available in the literature. Comparison between the results of the present method and these existing results are presented in Figs. 2 and 3. In the case of a simply supported beam, each Fourier sine term represents the mode with rotational spring constant $\beta = 0$. The present result for very small β is compared with the existing result from the literature.

In the absence of axial load, $P = 0$, the result is compared with the results reported by Woinowsky-Krieger. Results with axial load are compared with Burgreen's result. The present results from this paper are in the form of circle marks and Burgreen's and Woinowsky-Krieger's results are in the form of solid lines. The comparison shows that the results from the series expansion are in good agreement with the existing results.

3.2. Comparisons of the nonlinear frequencies for different boundary conditions

The comparison of nonlinear frequencies for beams with different end restraints, β, and zero axial load as a function of vibration amplitude, a/r, are presented in Table 1. The nonlinear frequency is nondimensionalized with respect to its corresponding linear frequency, $\tilde{\omega}/\omega_0$.

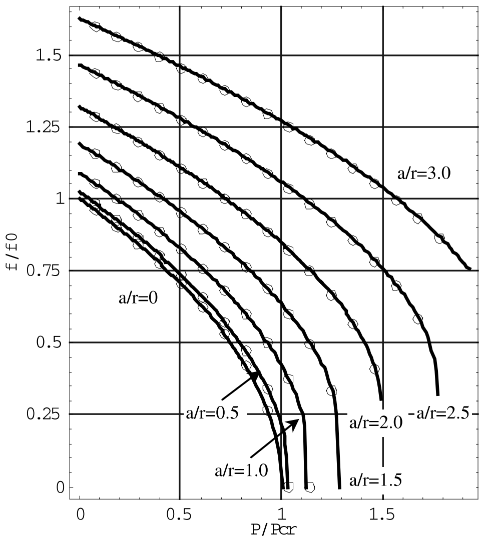

Fig. 2. Comparison between the present results and the existing results of simply supported beam, for the variation of frequency with axial load, P/P_{cr}.

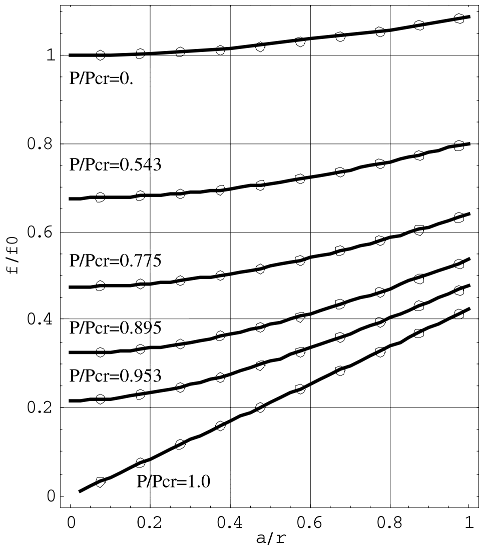

Fig. 3. Comparison between the present results and the existing results of simply supported beam, for the variation of frequency with amplitude vibration, a/r.

From the table, we can see that the frequency for fixed–fixed beam by using conventional mode shape (Section 2.2) and series expansion mode shape are in good agreement. Also, it is clearly seen that the effect of nonlinearity is more dominant for the beams with lower rotational spring constant, β.

3.3. Phase trajectory plot

The effect of the nonlinearity on the vibrations of a buckled beam is easier to notice in the phase trajectory plot, which plots the displacement, w, with respect to the velocity, $\dot{w}$ at a selected location. The phase plot for pre-buckling stage, for different values of the amplitude of deflections, is presented in Fig. 4. The phase plot corresponding to the post-buckled state of the beam, for different values of the amplitude of deflections, is presented in Fig. 5. The plots represent the trajectory of the point at $x = 0.5L$ and with the boundary conditions $\beta_0 = 10^5$ and $\beta_1 = 10$. In the pre-buckled stage, the trajectory of the motions of nonlinear system with small amplitude is a closed symmetric curve with a single center (fixed point). The

Table 1
Linear and nonlinear frequencies of the beams with different end restraints, β, and vibration amplitudes, a/r

β_0	β_1	ω_0 (Hz)	$\tilde{\omega}/\omega_0$ for $a/r = 0.5$	$\tilde{\omega}/\omega_0$ for $a/r = 1$	$\tilde{\omega}/\omega_0$ for $a/r = 1.5$
Fixed–fixed[a]		138.121	1.0056	1.0221	1.0492
Fixed–fixed[b]		139.258	1.0056	1.0223	1.0493
10^5	10^5	139.228	1.0056	1.0223	1.0494
10^5	10^3	137.808	1.0057	1.0225	1.0499
10^5	10^1	105.339	1.0090	1.0354	1.0777
10^3	10^3	136.409	1.0057	1.0227	1.0504
10^3	10^1	104.304	1.0091	1.0357	1.0785
0.1	0.1	61.166	1.0229	1.0885	1.1889
Simply supported[a]		60.6296	1.023	1.089	1.190

[a] Using conventional mode shapes.
[b] Using Fourier series mode shapes.

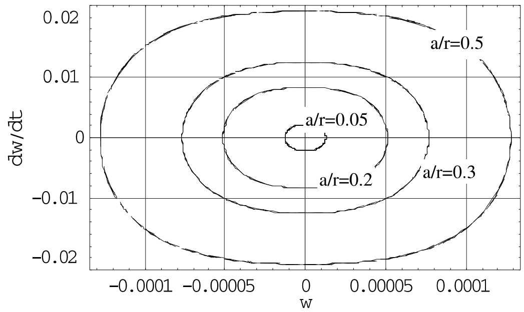

Fig. 4. Phase trajectory plot for pre-buckling stage, $P/P_{cr} = 0.83$, for point $x = 0.5L$.

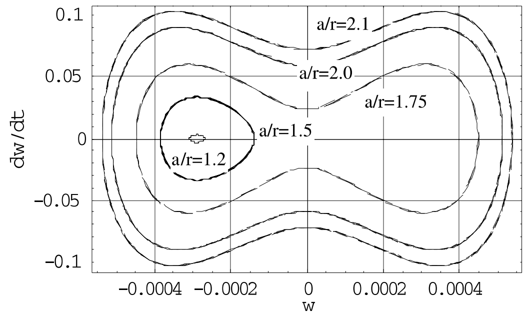

Fig. 5. Phase trajectory plot for post-buckling stage, $P/P_{cr} = 1.14$, for point $x = 0.5L$.

curve is more like an ellipse, while the linear system has a circular phase plot. In the post-buckled stage, the phase diagram has also a closed trajectory. The large amplitude of the trajectory is not a merely scaled-up version of those for small amplitude (Fig. 5). The phase plot really shows that the period of the motions of the nonlinear system depends on the amplitude. As the amplitude grows, the phase plot shapes change from a closed curve with one center into a closed curve with two centers (three fixed points). The motion is still harmonic and the phase trajectory encloses all three fixed points.

3.4. Comparison of results from different types of solutions presented in this paper

The comparison, between the use of the conventional mode shapes (Eq. (7)) and the use of series expansion (Eq. (31)), for a fixed–fixed beam results for the nonlinear vibration at the pre-buckling as well as post buckling conditions, are presented in Figs. 6 and 7. The variations of frequencies with axial load for different values of amplitude vibrations are shown in Fig. 6. The variations of frequencies with the amplitude of vibrations for different values of axial loads are shown in Fig. 7. It can be observed that the results using series expansion are in good agreement with the result from the use of conventional mode shapes. The presented results indicate that the results are on top of each other for most cases.

The plots (Fig. 7) of the pre-buckled state and the post-buckled state are separated by a straight line that represents the case when the axial load is equal to the buckling load. This can be verified as follows: At $P = P_{cr}$, the first and the third terms of Eq. (1) cancel each other for this free vibrations case ($F = 0$). Then, Eq. (12) reduces to

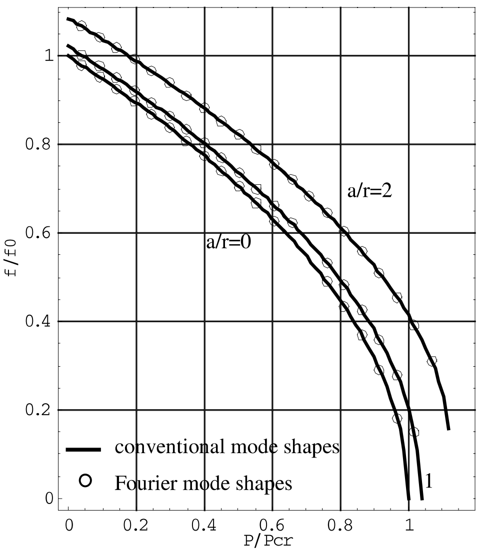

Fig. 6. Comparison of natural frequencies between the method that uses series expansion and that which uses conventional modes as function of axial load, P/P_{cr}, for different amplitude ratio a/r.

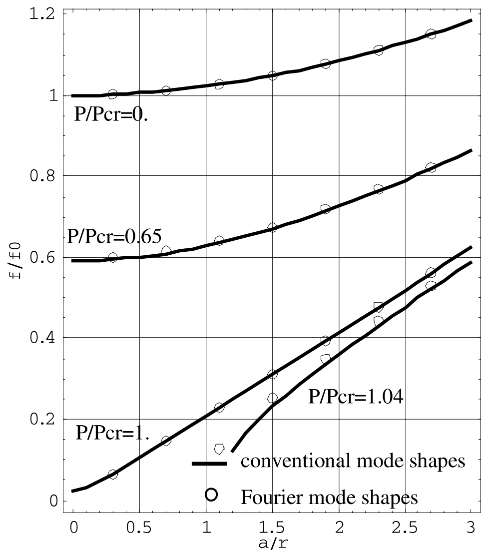

Fig. 7. Comparison of natural frequencies between the method that uses series expansion and that which uses conventional modes as function of amplitude vibration, a/r, for different axial loads, P/P_{cr}.

$$\left(\frac{d\psi}{d(pt)}\right)^2 = k^2(1-\psi^2)(1+\psi^2), \tag{37}$$

where $k^2 = 1/2$ and $p^2 = \chi_2$ as defined in Section 2.2.

Then, we can obtain elliptic integral in the form of

$$pt = 2\int_0^{\varphi} \frac{d\varphi}{\sqrt{(1+\sin^2\varphi)}}. \tag{38}$$

Since k^2 is not a function of the vibration amplitude, the period of the oscillation, T, does not depend on $\hat{a}$.

$$4T = 4\int_0^{\varphi} \frac{2d\varphi}{\sqrt{(1+\sin^2\varphi)}}. \tag{39}$$

Then, for simply supported boundary conditions, a linear relationship between the frequency, ω, and the deflection amplitude, $\hat{a}$, can be expressed as follows:

$$\omega = \frac{2\pi p}{T} \quad \text{or} \quad \omega = \frac{\pi}{T}\frac{\hat{a}}{r}\sqrt{\frac{EI}{\mu}}\left(\frac{\pi}{L}\right)^2. \tag{40}$$

This argument is also valid for any other boundary conditions, as shown in the figures of Section 3.5.

3.5. Nonlinear frequencies for different end restraints

The results for various end restraints are presented in Figs. 8–17. The variations of the frequencies with respect to the axial load and amplitude vibration are presented for each set of end restraints. The value of end restraint in the real structure depends on the connection between the structure supports. Usually, we simplify the end restraint into several standard conditions, such as simple support, fixed support or free support. But in practice, the end restraints are not ideal and can be considered to have elastic restraints. In the ideally fixed end case, the slope at the end is equal to zero and the value of β is infinite. When such a structure is built in practice the end restraints are not ideal. To model a practical structure, we need to take finite values of the rotational restraint β, depending on the connection conditions of the beam to the support.

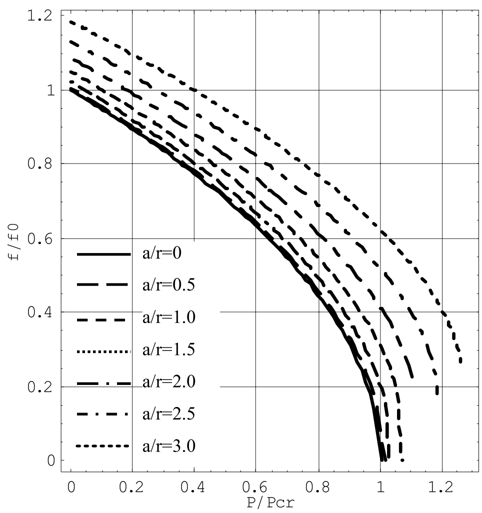

Fig. 8. Variation of frequencies with axial load, P/P_{cr}, for different amplitudes of vibrations, a/r, for $\beta_0 = 10^5$ and $\beta_1 = 10^5$.

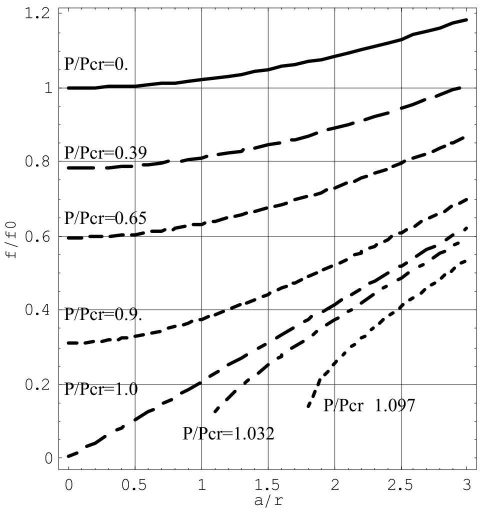

Fig. 9. Variation of frequencies with amplitude vibration, a/r, for different axial loads, P/P_{cr}, for $\beta_0 = 10^5$ and $\beta_1 = 10^5$.

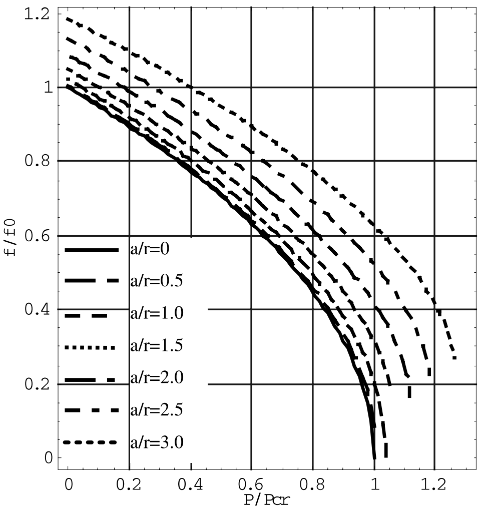

Fig. 10. Variation of frequencies with axial loads, P/P_{cr}, for different amplitudes of vibrations, a/r, for $\beta_0 = 10^5$ and $\beta_1 = 10^3$.

The ideal fixed end condition is realized when $\beta \to \infty$. However, in the numerical example, the value of the rotational restraint β was taken to be equal to $\beta = 10^5$ to simulate the ideally fixed end condition. For practical purposes, this value is large enough to simulate a fixed end condition. The case, $\beta = 10^3$ and $\beta = 10$, refers to a case in which the two support conditions are neither ideally fixed nor simply supported. The end, with $\beta = 10^3$ is closer to the fixed condition and $\beta = 10$ corresponds to a boundary that is closer to a simply supported boundary.

The changes of the nonlinear effects, as one of the end restraints becomes weaker ($\beta_0 = 10^5$ and β_1 varies from 10^5 to 0), are shown in Figs. 8–13. As the end restraint becomes weaker, i.e. the rotational restraint β becomes smaller; the effect of nonlinearity in the vibration is more significant. The contribution of the

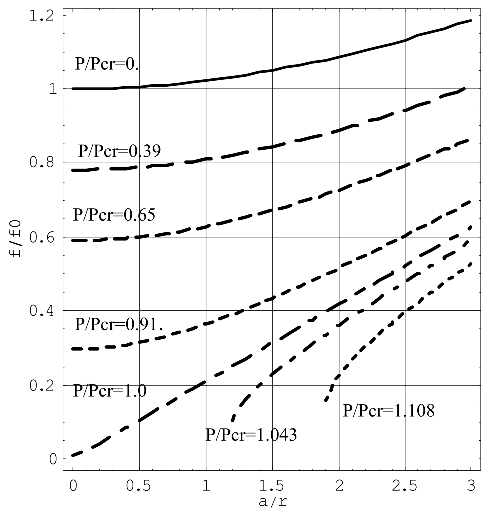

Fig. 11. Variation of frequencies with amplitudes of vibration, a/r, for different axial loads, P/P_{cr}, for $\beta_0 = 10^5$ and $\beta_1 = 10^3$.

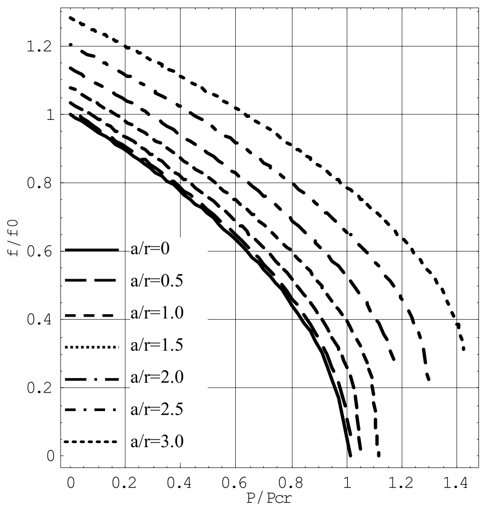

Fig. 12. Variation of frequencies with axial load, P/P_{cr}, for different amplitudes of vibrations, a/r, for $\beta_0 = 10^5$ and $\beta_1 = 10$.

amplitude of the vibration to nonlinear effects is clearly seen in Figs. 9, 11and 13. As the amplitude of the vibration, a/r, becomes larger, the nonlinear natural frequency of the beam with weak end restraints, small β, increases faster.

With the same axial load and the vibration amplitude the nonlinear frequency (f/f_0) of the beam with weak end restraints is lower than the one with stiffer restraints. The same phenomenon also shows in Figs. 14–17, where the value of one end restraint that is close to fixed end support is smaller, $\beta_0 = 10^3$. Comparing results of Figs. 8 and 9 with $\beta_0 = \beta_1 = 10^5$ and Figs. 14 and 15 with $\beta_0 = \beta_1 = 10^3$ we can verify that the end restraints with value of $\beta = 10^3$ are close to a fixed end support. The changes of the nonlinear frequency (f/f_0), as function of axial load, can be seen in Figs. 8, 10, 12, 14 and 16, for different values of

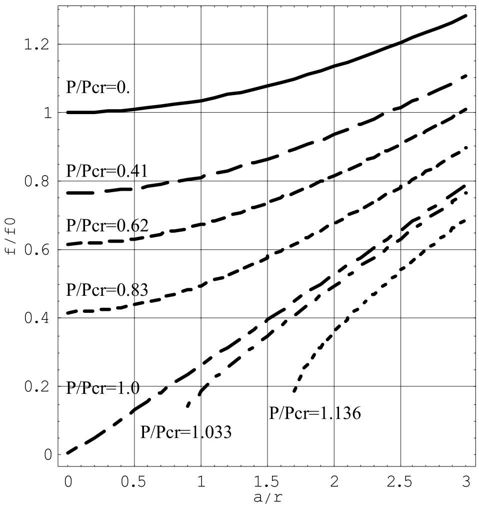

Fig. 13. Variation of frequencies with amplitudes of vibration, a/r, for different axial loads, P/P_{cr}, for $\beta_0 = 10^5$ and $\beta_1 = 10$.

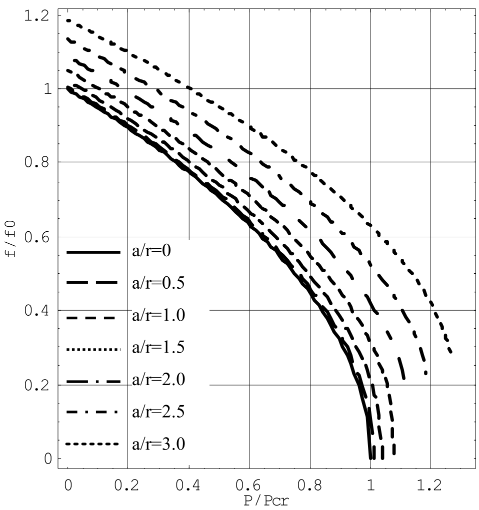

Fig. 14. Variation of frequencies with axial load, P/P_{cr}, for different amplitudes of vibrations, a/r, for $\beta_0 = 10^3$ and $\beta_1 = 10^3$.

end restraints. If the neutral axial of the beam were free to expand in the axial direction, the axial stretching term is not present in the differential equation. However, we can still have an axial stretching if we have axial spring instead of an ideally free axial restraint.

4. Conclusions

Exact solutions for buckled beams with various rotational end restraints are obtained by using modes from the linear theory. These linear modes readily satisfy the boundary conditions. The nonlinear solutions

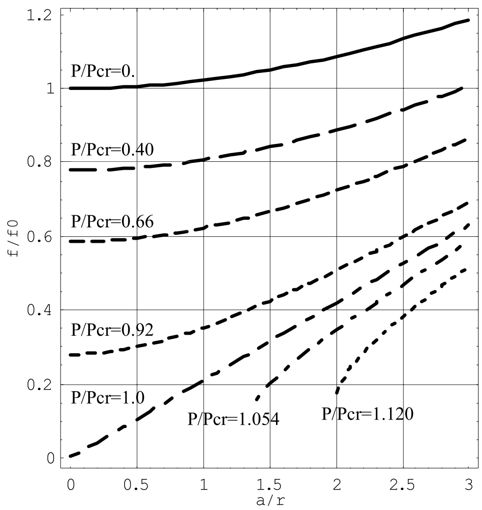

Fig. 15. Variation of frequencies with amplitudes of vibration, a/r, for different axial loads, P/P_{cr}, for $\beta_0 = 10^3$ and $\beta_1 = 10^3$.

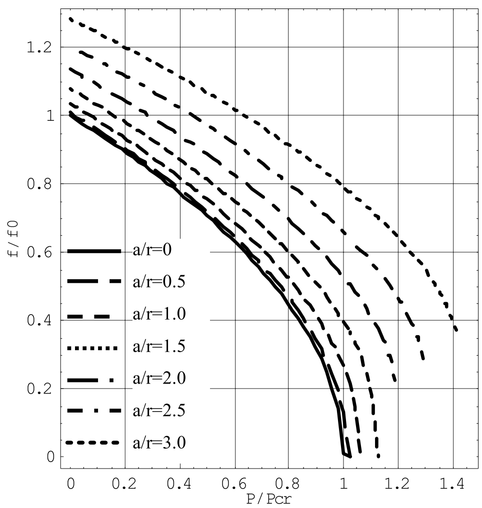

Fig. 16. Variation of frequencies with axial load, P/P_{cr}, for different amplitudes of vibrations, a/r, for $\beta_0 = 10^3$ and $\beta_1 = 10$.

for the buckled beam are then obtained by using elliptic functions. The nonlinear natural frequency of a beam with a constant axial load or with a constant distance between its end restraints increases with the amplitude of vibration. This frequency is higher than those predicted by the linear theory.

The effect of the axial load and the vibration amplitude are less dominant for beams with stiffer end restraints or large rotational spring constants. When the axial load is a compressive force in the beam and is exactly equal to the critical buckling load, the theoretical results indicate that the frequency varies linearly with the amplitude of vibration. The results for a fixed–fixed and simply supported beam for the nonlinear solutions using series expansion are in good agreement with the result from the use of conventional mode shapes using trigonometric and hyperbolic functions.

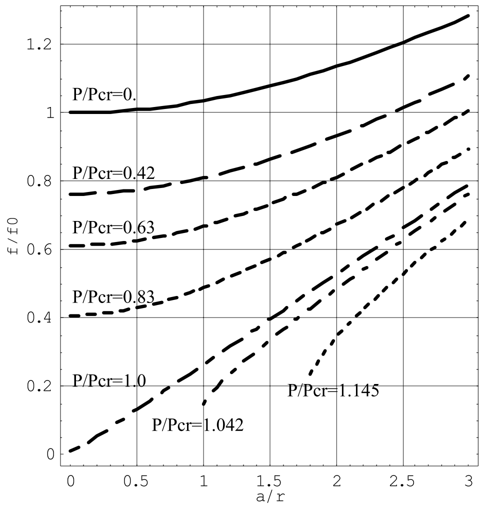

Fig. 17. Variation of frequencies with amplitudes of vibration, a/r, for different axial loads, P/P_{cr}, for $\beta_0 = 10^3$ and $\beta_1 = 10$.

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